Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 51-32 (2018) 434–438

Solution of Inverse Problems for Control Systems with Large Control Parameter Dimension ⋆

Evgeniy A. Krupennikov *

* *N.N. Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences (IMM UB RAS) 16 S.Kovalevskaya str., Yekaterinburg, 620990 Russia; Ural Federal University named after the first President of Russia B.N.Yeltsin (UrFU), 620002, 19 Mira str., Ekaterinburg, Russia (e-mail: krupennikov@imm.uran.ru)*

Abstract: The a posteriori analysis of the realized motions (in other words, trajectories and controls) is an important part of the theory of optimal control and decision making.
This paper is devoted to solving inverse problems of reconstruction of realized controls for control systems using known inaccurate measurements of the realized trajectories.
A new method for solving inverse problems is suggested for a class of control systems with dynamics linear in controls and non-linear in state coordinates where the dimension of the control parameter is greater than or equal to the dimension of the state variable. This method relies on necessary optimality conditions in auxiliary variational problems.
An illustrating example is exposed.

Keywords: Nonlinear control systems, inverse problem, calculus of variations, Hamiltonian systems.

1. INTRODUCTION

This paper is devoted to solving inverse problems of reconstruction of controls for control systems using known inaccurate measurements of the realized trajectories.

Inverse problems may occur in many areas such as economics, engineering, medicine and many others that involve the task of reconstruction of the controls by known inaccurate trajectory measurements.

The inverse problems have been studied by many authors. The approach suggested by Kryazhimskii and Osipov (1984); Osipov and Kryazhimskii (1995) is one of the closest to the material of this paper. The method suggested by A. V. Kryazhimskii and Yu. S. Osipov reconstructs the controls by using a regularized (a variation of Tikhonov regularization, see Tikhonov (1943)) procedure of control with a guide. This procedure allows to reconstruct the controls on-line. It is originated from the works of Krasovskii's school on the theory of optimal feedback control, see Krasovskii (1968); Krasovskii and Subbotin (1974).

Another method for solving dynamic reconstruction problems by known history of inaccurate measurements has been suggested by Subbotina et al. (2015). It is based on a method, which uses necessary optimality conditions for auxiliary optimal control problems. This method has been also developed by Subbotina and Krupennikov (2016); Subbotina et al. (2017); Subbotina and Krupennikov (2017); Krupennikov (2018). A modification of this approach is presented in this paper. It relies on necessary optimality conditions for auxiliary variational problems on extremum for an integral functional. The functional is a variation of a Tikhonov regularizator.

In this paper we consider a class of control systems with dynamics linear in controls and non-linear in state coordinates where the dimension of the control parameter is greater than or equal to the dimension of the state variable.

Results of a simulation are exposed as an example.

2. DYNAMICS

We consider control systems with dynamics of the form

$$\begin{gathered}\dot{x}(t) = G(x(t),t)u(t),\\ x(\cdot):[0,T]\to\mathbb{R}^n,\quad u(\cdot):[0,T]\to\mathbb{R}^m,\\ m\geq n,\quad t\in[0,T].\end{gathered} \tag{1}$$

Here $G(x,t)$ is an $n\times m$ matrix with elements $g_{ij}(x,t):\mathbb{R}^n\times[0,T]\to\mathbb{R},\ i=1,...,n,\ j=1,...,m$ that have continuous derivatives

$$\begin{gathered}\frac{\partial g_{ij}(x,t)}{\partial t},\quad \frac{\partial g_{ij}(x,t)}{\partial x_k},\\ i=1,...,n,\quad j=1,...,m,\quad k=1,...,n,\\ x\in\mathbb{R}^n,\quad t\in[0,T].\end{gathered} \tag{2}$$

In (1) $x(t)$ is the state coordinates vector and $u(t)$ is the controls vector. The admissible controls are piecewise

⋆ This work was supported by the Russian Foundation for Basic Research (project no. 17-01-00074) and by the Ural Branch of the Russian Academy of Sciences (project no. 18-1-1-10).

10.1016/j.ifacol.2018.11.423

continuous functions with finite number of points of discontinuity satisfying

$$u(t) \in U, \quad t \in [0,T], \tag{3}$$

where $U \subset \mathbb{R}^n$ is a convex compact set.

3. INPUT DATA

It is supposed that some base trajectory $x^*(\cdot) : [0,T] \to \mathbb{R}^n$ of system (1) has been generated by an admissible control. The properties of dynamics (1) provide (see Subbotina et al. (2015)) that there exists a unique admissible normal control $u^*(\cdot)$ — the admissible control that generates the trajectory $x^*(\cdot)$ and has the least possible norm in L_2 space

$$\begin{aligned} \|u^*(\cdot)\|_{L^2[0,T]} = \sqrt{\int_0^T \|u^*(t)\|^2 dt} \\ = \min_{u(\cdot):[0,T]\to\mathbb{R}^m:\ \dot{x}^*(t)=G(x^*(t),t)u(t)} \|u(\cdot)\|_{L^2[0,T]}. \end{aligned} \tag{4}$$

Hereinafter $\|f\| = \sqrt{\sum_{i=1}^{k} f_i^2}$, $f \in \mathbb{R}^k$, $k \in \mathbb{N}$ is Euclidean norm in $\mathbb{R}^k$.

We assume that inaccurate measurements $y(\cdot,\delta) = y^\delta(\cdot) : [0,T] \to \mathbb{R}^n$ of the base trajectory $x^*(\cdot)$ are known and are twice continuously differentiable functions that determine $x^*(t)$ with the known accuracy $\delta > 0$,

$$\|y^\delta(\cdot) - x^*(\cdot)\|_{C[0,T]} \le \delta. \tag{5}$$

Hereinafter

$$\begin{aligned} \|f(\cdot)\|_{C[0,T]} = \max_{t\in[0,T]} \|f(t)\|, \\ f(\cdot) : [0,T] \to \mathbb{R}^k, \quad k \in \mathbb{N} \end{aligned} \tag{6}$$

is the norm in the space of continuous functions C.

4. HYPOTHESES

We introduce two hypotheses on the input data.

Hypothesis 1. There exist compact set $\Psi \subset \mathbb{R}^n$ and constant $r > 0$ such that

$$\Psi \supset \{x \in \mathbb{R}^n : ||x - x^*(t)|| \le r,\ \forall t \in [0,T]\} \tag{7}$$

and rows of the matrix $G(x,t)$ are linearly independent for $(x,t) \in \Psi \times [0,T]$.

Hypothesis 2. There exist constants $\delta_0 > 0$ and $\overline{Y} > 0$ such that for any $\delta \in (0, \delta_0]$

$$\begin{gathered} |y_i^\delta(t)| \le \overline{Y}, \quad |\dot{y}_i^\delta(t)| \le \overline{Y}, \quad |\dot{x}_i^*(t)| \le \overline{Y}, \\ t \in [0,T], \quad i = 1,\dots,n \end{gathered} \tag{8}$$

and for any $\delta \in (0,\delta_0]$ exists compact set $\Omega^\delta \subset [0,T]$ with measure $\mu\Omega^\delta = \beta^\delta \overset{\delta\to 0}{\longrightarrow} 0$ such that

$$\begin{gathered} |\ddot{y}_i^\delta(t)| \le \overline{Y}, \quad t \in [0,T] \setminus \Omega^\delta, \\ \max_{t\in\Omega^\delta} |\ddot{y}_i^\delta(t)|\beta^\delta \le \overline{Y}, \\ i = 1,\dots,n. \end{gathered} \tag{9}$$

Remark 3. Conditions (8) reflect the fact that the right-hand side of equation (1) is restricted.

Remark 4. In hypothesis 2 the constant $\overline{Y}$ is unified for all inequalities to simplify the further calculations and explanations.

Remark 5. Hypothesis 2 allows the functions $\dot{y}^\delta(\cdot)$ to be able to approximate piecewise continuous functions $\dot{x}^*(\cdot) = G(x^*(\cdot),\cdot)u^*(\cdot)$.

5. PROBLEM STATEMENT

Let us consider the following inverse problem: for a given $\delta \in (0,\delta_0]$ and a given measurement $y^\delta(\cdot)$ satisfying inequalities (5) and Hypothesis 2 to find a function $u(\cdot,\delta) = u^\delta(\cdot) : [0,T] \to \mathbb{R}^m$ that satisfies the following conditions:

C1 The function $u^\delta(\cdot)$ belongs to the set of admissible controls, i.e. the set of piecewise continuous functions with finite number of points of discontinuity satisfying constraints (3).

C2 The control $u^\delta(\cdot)$ generates a trajectory $x(\cdot,\delta) = x^\delta(\cdot) : [0,T] \to \mathbb{R}^n$ of system (1) with boundary condition $x^\delta(T) = y^\delta(T)$.

C3 The functions $x^\delta(\cdot)$ and $u^\delta(\cdot)$ satisfy conditions

$$\begin{aligned} \lim_{\delta\to 0} \|x^\delta(\cdot) - x^*(\cdot)\|_{C[0,T]} = 0, \\ \lim_{\delta\to 0} \|u^\delta(\cdot) - u^*(\cdot)\|_{L^2[0,T]} = 0. \end{aligned} \tag{10}$$

6. SOLUTION OF THE INVERSE PROBLEM

6.1 Auxilliary problem

To solve inverse problem **C1**–**C3**, we introduce an auxiliary variational problem (AVP) for fixed parameters $\delta \in (0,\delta_0]$, $\alpha > 0$ and a given measurement $y^\delta(\cdot)$ satisfying inequalities (5) and Hypothesis 2.

We consider the set of pairs of continuously differentiable functions $F_{xu} = \{\{x(\cdot),u(\cdot)\} : x(\cdot) : [0,T] \to \mathbb{R}^n, \quad u(\cdot) : [0,T] \to \mathbb{R}^m\}$ that satisfy differential equations (1) and the following boundary conditions

$$\begin{aligned} x(T) = y^\delta(T), \quad u(T) = G^T(y^\delta(T),T) \\ \cdot\Big(G(y^\delta(T),T)G^T(y^\delta(T),T)\Big)^{-1} \dot{y}^\delta(T). \end{aligned} \tag{11}$$

Hereinafter A^{-1} is the inverse of a non degenerate square matrix A. Let us remark that due to hypothesis 1, the inverse $\Big(G(y^\delta(T),T)G^T(y^\delta(T),T)\Big)^{-1}$ exists.

AVP is to find a pair of functions $x(\cdot,\delta,\alpha) = x^{\delta,\alpha}(\cdot) : [0,T] \to \mathbb{R}^n$ and $u(\cdot,\delta,\alpha) = u^{\delta,\alpha}(\cdot) : [0,T] \to \mathbb{R}^m$ such that $\{x^{\delta,\alpha}(\cdot), u^{\delta,\alpha}(\cdot)\} \in F_{xu}$ and such that they provide an extremum (minimum) for the integral functional

$$I(x(\cdot),u(\cdot)) = \int_0^T \left[-\frac{\|x(t) - y^\delta(t)\|^2}{2} + \frac{\alpha^2\|u(t)\|^2}{2} \right] dt. \tag{12}$$

Here α is a small regularising (see Tikhonov (1943)) parameter.

6.2 Necessary optimality conditions for the AVP

We can write the necessary optimality conditions for AVP (1),(12),(11) in Lagrange form (see Ioffe and Tikhomirov (1974)). Lagrangian for the AVP has the form

$$L(x,u,\dot{x},\lambda,t) = -\frac{\|x-y^\delta(t)\|^2}{2} + \frac{\alpha^2\|u\|^2}{2} + \left\langle \lambda^T, \dot{x} - G(x,t)u \right\rangle, \tag{13}$$

where λ is the Lagrange multipliers vector and $\langle\cdot,\cdot\rangle$ is the scalar vector multiplication.

The $n+m$ corresponding Euler equations are

$$\begin{gathered}\dot{\lambda}_i(t) + (x_i(t) - y_i^\delta(t)) \\ + \sum_{j=1}^{n}\left[\lambda_j(t)\sum_{k=1}^{m}\frac{\partial g_{jk}}{\partial x_i}(x(t),t)u_k(t)\right] = 0, \\ i = 1,\ldots,n, \\ -\alpha^2 u_h(t) + \sum_{j=1}^{n}[\lambda_j(t) g_{jh}(x(t),t)] = 0, \quad h = 1,\ldots,m.\end{gathered} \tag{14}$$

The last m equations in (14) define the relations between the controls $u(t)$ and the Lagrange multipliers $\lambda(t)$:

$$u(t) = \frac{1}{\alpha^2}G^T(x(t),t)\lambda(t). \tag{15}$$

We can substitute equations (15) into (14) and (1) to rewrite them in the form of Hamiltonian equations, where the vector $s(t) = -\lambda(t)$ plays the role of the adjoint variables vector:

$$\begin{aligned}&\dot{x}(t) = -(1/\alpha^2)G(x(t),t)G^T(x(t),t)s(t), \\ &\dot{s}_i(t) = x_i(t) - y_i^\delta(t) \\ &+\frac{1}{\alpha^2}\langle s(t), \frac{\partial G}{\partial x_i}(x(t),t)G^T(x(t),t)s(t)\rangle, \quad i = 1,\ldots,n.\end{aligned} \tag{16}$$

Here $\frac{\partial G}{\partial x_i}(x(t),t)$ is a matrix with elements $\frac{\partial g_{jk}}{\partial x_i}(x(t),t)$, $j = 1,\ldots,n$, $k = 1,\ldots,m$.

By substituting (15) into (11), one can obtain the boundary conditions for system (16):

$$\begin{aligned}&x(T) = y^\delta(T), \\ &s(T) = -\alpha^2\big(G(y^\delta(T),T)G^T(y^\delta(T),T)\big)^{-1}\dot{y}^\delta(T).\end{aligned} \tag{17}$$

Thus, we have got the necessary optimality conditions for the AVP (1),(12),(11) in Hamiltonian form (16),(17).

Remark 6. The suggested algorithm for finding the solution of inverse problem **C1**–**C3** utilizes only necessary conditions (16),(17) which provide a stationary point for functional (12) irrespectively of whether the extremum is reached. Thus, it is not verified if an extremum is actually reached in the AVP.

6.3 Solution of the inverse problem

Let us consider the function

$$u^{\delta,\alpha}(\cdot) = -(1/\alpha^2)G^T(x^{\delta,\alpha}(\cdot),\cdot)s^{\delta,\alpha}(\cdot), \tag{18}$$

where $x^{\delta,\alpha}(\cdot)$, $s^{\delta,\alpha}(\cdot)$ are the solutions of system (16) with boundary conditions (17).

We will consider the cut-off functions

$$\hat{u}^{\delta,\alpha}(t) = \begin{cases} u^{\delta,\alpha}(t) & , \quad u^{\delta,\alpha}(t) \in U, \\ \hat{u} \in \mathbb{R}^m : \|u^{\delta,\alpha}(t) - \hat{u}\| & \\ \quad = \min\limits_{u\in U}\|u^{\delta,\alpha}(t) - u\| & , \quad u^{\delta,\alpha}(t) \notin U. \end{cases} \tag{19}$$

as a basis for a solution of inverse problem **C1**–**C3**.

6.4 Convergence of the solution

System (16) can be written in the form

$$\dot{z}(t) = B(x(t),t)z(t) + F(z(t),t). \tag{20}$$

Here

$$\begin{gathered} z(\cdot) = (x_1(\cdot), x_2(\cdot), \ldots, x_n(\cdot), s_1(\cdot), s_2(\cdot), \ldots, s_n(\cdot)), \\ B(x,t) = \begin{pmatrix} O_n & G(x,t)G^T(x,t) \\ I_n & O_n \end{pmatrix}, \\ F(z(t),t) = \Big(\underbrace{0,\ldots,0}_{n}, -y_1^\delta(t) \\ +\frac{1}{\alpha^2}\langle s(t), \frac{\partial G}{\partial x_1}(x(t),t)G^T(x(t),t)s(t)\rangle, \ldots, -y_n^\delta(t) + \\ \frac{1}{\alpha^2}\langle s(t), \frac{\partial G}{\partial x_n}(x(t),t)G^T(x(t),t)s(t)\rangle\Big), \end{gathered} \tag{21}$$

where O_n is an $n\times n$ zero matrix and I_n is an $n\times n$ identity matrix.

Hypothesis 1 and the fact that for any non-singular matrix A the matrix AA^T is positive definite provide that for $x \in \Psi$, $t \in [0,T]$ the eigenvalues of the matrix $B(x,t)$ are imaginary. From this fact and the results of Krupennikov (2018) follows the validity of the following theorems

Theorem 7. For a fixed $\delta \in (0,\delta_0]$ there exists a parameter $\alpha(\delta) \xrightarrow{\delta\to 0} 0$ such that for any fixed $\alpha \le \alpha(\delta)$ the solution $x^{\delta,\alpha}(\cdot)$, $s^{\delta,\alpha}(\cdot)$ of system (16) with boundary conditions (17) exists and is unique and extandable on $[0,T]$ and $x^{\delta,\alpha}(t) \in \Psi$ for any $t \in [0,T]$ (where the set Ψ is from hypothesis 1).

Let us now consider for any fixed $\tilde{\delta} \in (0,\delta_0]$ and $\tilde{\alpha} \le \alpha(\tilde{\delta})$ functions $\tilde{u}^{\delta,\alpha}(\cdot)$ — the cut-off function (19) constructed for

$$u^{\tilde{\delta},\tilde{\alpha}}(\cdot) = -(1/\tilde{\alpha}^2)G^T(x^{\tilde{\delta},\tilde{\alpha}}(\cdot),\cdot)s^{\tilde{\delta},\tilde{\alpha}}(\cdot), \tag{22}$$

where $x^{\tilde{\delta},\tilde{\alpha}}(\cdot)$, $s^{\tilde{\delta},\tilde{\alpha}}(\cdot)$ are the solutions of system (16) with boundary conditions (17) for $\delta = \tilde{\delta}$, $\alpha = \tilde{\alpha}$.

Theorem 8. If Hypotheses 1,2 are true then the functions $\tilde{u}^{\delta,\alpha}(\cdot)$ satisfy conditions **C1**–**C3**.

Theorem 8 means that the functions $\tilde{u}^{\delta,\alpha}(\cdot)$ are a solution of inverse problem **C1**–**C3**.

7. REMARKS ON THE SUGGESTED METHOD

Let us now consider another approach to inverse problem **C1**–**C3** and compare it with the approach suggested in this paper.

We consider the functions

$$u^\delta(\cdot) = G^g(y^\delta(\cdot),\cdot)\dot{y}^\delta(\cdot), \tag{23}$$

where $G^g(\cdot) = G^T(\cdot)\Big(G(\cdot)G^T(\cdot)\Big)^{-1}$ is the generalized inverse of the matrix $G(\cdot)$.

We now consider functions $\hat{u}^{\delta}(t)$ which are the cut-off functions (19) constructed for functions (23).

Hypotheses 1,2 provide in particular that

$$\|\dot{y}^{\delta}(\cdot) - \dot{x}^{*}(\cdot)\|_{L^2[0,T]} \xrightarrow{\delta\to 0} 0. \tag{24}$$

Convergence (24) and inequalities (5),in turn, provide that

$$\|\hat{u}^{\delta}(\cdot) - u^{*}(\cdot)\|_{L^2[0,T]} \xrightarrow{\delta\to 0} 0. \tag{25}$$

So, the functions $\hat{u}^{\delta}(t)$ solve inverse problem **C1–C3**.

Remark 9. Numerical realisation of this approach involves the problem of finding inverse matrix $\Big(G(y^{\delta}(t),t)G^{T}(y^{\delta}(t),t)\Big)^{-1}$ for $t \in [0,T]$.

Comparing approach (23) and the approach suggested in this paper, we can see that the second one reduces the task of finding a generalized inverse of a variable $n \times m$ matrix $G(y^{\delta}(t),t)$ to the task of solving a system of $2n$ ODEs.

In some applications numerical integration of ODE systems may be more preferable than matrix inversing.

8. EXAMPLE

To illustrate the application of the suggested method let us consider the following control system

$$\begin{gathered}\dot{x}(t) = G(x(t),t)u(t),\\ x(\cdot) : [0,3] \to \mathbb{R}^2,\ u(\cdot) : [0,3] \to \mathbb{R}^3,\\ G(x,t) = \begin{pmatrix} x_1^2 + t & 2\sqrt{t+1} & \sin x_2 \\ -3t+2 & \dfrac{x_1 x_2}{t+1} & 4 \end{pmatrix},\\ \|u(t)\| \le 3,\ t \in [0,3].\end{gathered} \tag{26}$$

We assume that some base trajectory $x^{*}(t)$ of system (26) has been realized on the interval $[0,3]$. We also assume that we know inaccurate measurements of the base trajectory which are twice continuously differentiable functions $y_1^{\delta}(t),\ y_2^{\delta}(t)$ fulfilling estimates (5) and Hypothesis 2.

Remark 10. To model inaccurate measurements random perturbations were applied to the base trajectories, assuming $x_1^{*}(t) = \sqrt{t+1} - 1,\ x_2^{*}(t) = t^2$. Hypothesis 1 is fulfilled for such functions and dynamics (26).

We consider inverse problem **C1–C3** for dynamics (26) and functions $y_1^{\delta}(t),\ y_2^{\delta}(t)$.

The trajectory $x^{\alpha,\delta}(t)$ and the control $\hat{u}^{\alpha,\delta}(t)$ were obtained numerically. The results are partly presented on pictures 1–2. The graphics of the first state variable $x_1^{\alpha,\delta}(t)$ and the third control $\hat{u}_3^{\alpha,\delta}(t)$ reconstructed for various values of parameters are shown. These components were chosen because their graphs are visually the most representative. On picture 2 $u_3^{*}(t)$ is the control, obtained by approach (23) discussed in section 7.

9. CONCLUSION

A new method for solving inverse problems of reconstruction of controls is presented and illustrated for a class of control systems with dynamics linear in controls and non-linear in state coordinates where the dimension of the control parameter is greater than or equal to the dimension of the state variable. The suggested approach may be considered as reducing the task of finding an inverse of a variable matrix to solving a system of ODEs.

Application of the suggested approach for the case of systems with the dimension of the control parameter less than the dimension of the state variable is a matter of the future research. Comparison of effectiveness of the method presented in this paper and of other know methods for solving dynamic reconstruction problems will also be presented in the future papers.

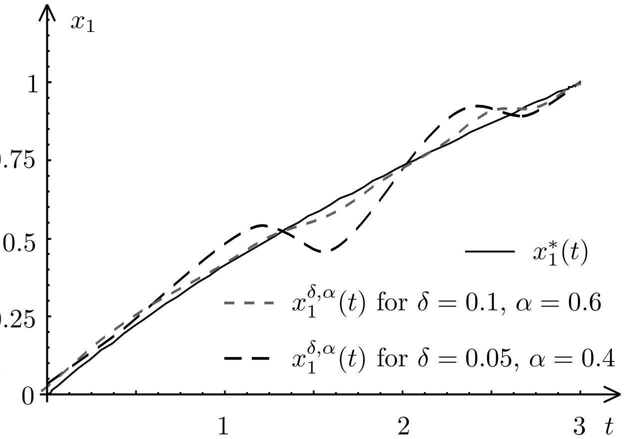

Fig. 1. Graphics of $x_1^{\delta,\alpha}(t),\ t \in [0,3]$ for various values of approximation parameters.

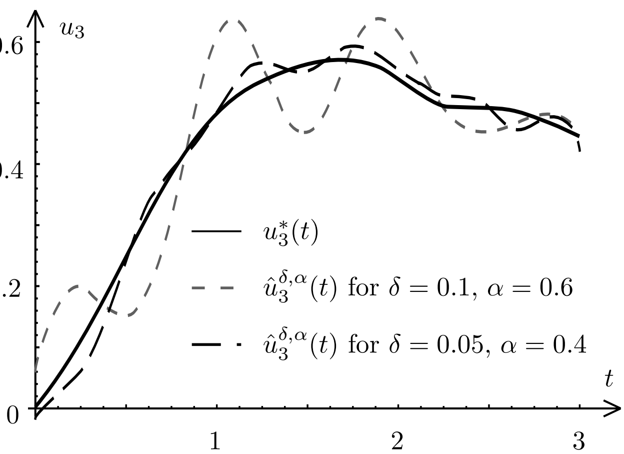

Fig. 2. Graphics of $\hat{u}_3^{\delta,\alpha}(t),\ t \in [0,3]$ for various values of approximation parameters.

ACKNOWLEDGEMENTS

This work was supported by the Russian Foundation for Basic Research (project no. 17-01-00074) and by the Ural Branch of the Russian Academy of Sciences (project no. 18-1-1-10).

REFERENCES

Ioffe, A. and Tikhomirov, V. (1974). *Teoriya extremalnih zadach [Theory of Extremum Problems]*. Nauka, Moscow. (in Russian).

Krasovskii, N.N. (1968). *Teoriya upravleniya dvizheniem [Theory of motion control]*. Nauka, Moscow. (in Russian).

Krasovskii, N.N. and Subbotin, A.I. (1974). *Positcionnie differentcialnie igri [Positional differential games].* Nauka, Moscow. (in Russian).

Krupennikov, E.A. (2018). On control reconstruction problems for dynamic systems linear in controls. In *Static & Dynamic Game Theory: Foundations & Applications*. Birkhauser. Unpublished.

Kryazhimskii, A.V. and Osipov, Y.S. (1984). Modelling of a control in a dynamic system. In *Engrg. Cybernetics*, 21(2), 38–47. Scripta Pub. Co., Washington.

Osipov, Y.S. and Kryazhimskii, A.V. (1995). *Inverse problems for ordinary differential equations: dynamical solutions*. Gordon and Breach, London.

Subbotina, N.N. and Krupennikov, E.A. (2016). Dynamic programming to identification problems. In *World J. Eng. Technol.*, 4(3B), 228–234.

Subbotina, N.N. and Krupennikov, E.A. (2017). The method of characteristics in an identification problem. In *Proceedings of the Steklov Institute of Mathematics*, 299, Suppl. 1, 205–216. Pleiades Publishing, Ltd.

Subbotina, N.N., Tokmantsev, T.B., and Krupennikov, E.A. (2015). On the solution of inverse problems of dynamics of linearly controlled systems by the negative discrepancy method. In *Proceedings of the Steklov Institute of Mathematics*, 291, 253–262. Pleiades Publishing, Ltd.

Subbotina, N.N., Tokmantsev, T.B., and Krupennikov, E.A. (2017). Dynamic programming to reconstruction problems for a macroeconomic model. In *IFIP Advances in Information and Communication Technology. [S.l.]*, 494, 472–481. Springer.

Tikhonov, A.N. (1943). Ob ustoichivosti obratnih zadach [on the stability of inverse problems]. In *Doklady Academii Nauk SSSR*, 39, 195–198. USSR Academy of Sciences, Moscow. (in Russian).